

Analytical solutions of the Schrödinger equation with the Manning-Rosen potential plus a Ring-Shaped like potential

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(Dated: October 22, 2012)

Abstract

The analytical solution of the Schrödinger equation for the Manning-Rosen potential plus a ring-shaped like potential is obtained by applying the Nikiforov-Uvarov method by using the improved approximation scheme to the centrifugal potential for arbitrary l states. The energy levels are worked out and the corresponding normalized eigenfunctions are obtained in terms of orthogonal polynomials for arbitrary l states.

PACS numbers: 03.65.Ge

Keywords: Nikiforov-Uvarov method, Manning-Rosen and Ring-Shaped potential

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I. INTRODUCTION

As known, one of the main objectives in theoretical physics since the early years of quantum mechanics (QM) is to obtain an exact solution of the Schrödinger equation (SE) for some special potentials of physical interest. Since the wave function contains all necessary information for full description of a quantum system, an analytical solution of the SE is of high importance in non-relativistic and relativistic quantum mechanics [1, 2]. There are few potentials for which the SE can be solved explicitly for all n and l quantum states.

The non-central potentials are needed to obtain better results than central potentials about the dynamical properties of the molecular structures and interactions. Researchers added ring shaped potentials to certain potentials, i.e, Coulomb, Kratzer [3] and Manning-Rosen potentials [4] to obtain non-central potentials. Ring-shaped potentials can be used in quantum chemistry to describe the ring shaped organic molecules such as benzene and in nuclear physics to investigate the interaction between deformed pair of nucleus and spin orbit coupling for the motion of the particle in the potential fields.

It would be interesting and important to solve the SE for the Manning-Rosen potential plus a Ring-Shaped like potential for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems. Thus, one can obtain the energy eigenvalues and corresponding eigenfunctions of the one particle problem within this potential. The central Manning-Rosen potential is defined by

$$V(r) = \frac{1}{kb^2} \left[\frac{\alpha(\alpha-1)\exp(-2r/b)}{(1-\exp(-r/b))^2} - \frac{A\exp(-r/b)}{(1-\exp(-r/b))} \right], \quad k = 2\mu/\hbar^2. \quad (1.1)$$

where A and α are dimensionless parameters, and b is the screening parameter. This potential is used as a mathematical modeling of the diatomic molecular vibrations and it constitutes a convenient model for other physical situations. It is known that, using for this potential the SE can be solved exactly for s-wave ($l = 0$) [5]. Unfortunately, for an arbitrary l -states ($l \neq 0$), the SE does not admit an exact solution. In such a case, the SE can be solved numerically or approximately using approximation schemes [6].

The potential which is solved in this study is obtained by adding ring-shaped potential term [7] as,

$$V(r, \theta) = \frac{1}{k} \left[\frac{\alpha(\alpha-1)\exp(-2r/b)}{b^2(1-\exp(-r/b))^2} - \frac{A\exp(-r/b)}{b^2(1-\exp(-r/b))} + \frac{\beta'}{r^2 \sin^2 \theta} + \frac{\beta \cos \theta}{r^2 \sin^2 \theta} \right]. \quad (1.2)$$

So far, many methods were developed and applied, such as supersymmetry (SUSY) [8, 9], factorization [10], Laplace transform approach [11] and the path integral method [12], to solve the radial SE exactly or quasi-exactly for $l \neq 0$ within these potentials. An other method known as the Nikiforov-Uvarov (NU) method [13] was proposed for solving the SE analytically. Many works show the power and simplicity of NU method in solving central and noncentral potentials [14–16]. This method is based on solving the second order linear differential equation by reducing to a generalized equation of hypergeometric type which is a second order homogeneous differential equation with polynomials coefficients of degree not exceeding the corresponding order of differentiation.

In this study, we obtain the energy eigenvalues and corresponding eigenfunctions for arbitrary l states by solving the SE for the Manning-Rosen potential plus a ring-shaped like potential using NU method. Moreover, by changing parameters we also obtain solutions for Hulthen potential [17], and Hulthen plus ring shaped potential. The NU method was also used to solve the SE for this potential. But our obtained results are more general and more efficient those given in Ref. [18]

The organization of this paper is as follows. The SE within Manning Rosen plus a ring-shaped like potential is provided in Section II. Bound state solution of the radial SE by NU method is presented in Section III. The solution of angle-dependent part of the SE is presented in Section IV and, the numerical results for energy levels and the corresponding normalized eigenfunctions are presented in Section V. Finally, some concluding remarks are stated in Section VI.

II. THE SCHRÖDINGER EQUATION WITH THE MANNING-ROSEN POTENTIAL PLUS A RING-SHAPED LIKE POTENTIAL

The Schrödinger equation in spherical coordinates is given as

$$\nabla^2 \psi + \frac{2\mu}{\hbar^2}[E - V(r, \theta)]\psi = 0. \quad (2.1)$$

Considering this equation, the total wave function is written as

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi), \quad (2.2)$$

where the polar angle solution is given by

$$\Phi(\phi) = \frac{1}{2\pi} e^{im\phi}, m = 0, \pm 1, \pm 2, \dots \quad (2.3)$$

Thus for radial and azimuthal SE for Manning Rosen plus ring-shaped like potential are

$$R''(r) + \frac{2}{r}R'(r) + \left[\frac{2\mu}{\hbar^2}E + \frac{1}{b^2} \frac{Ae^{-r/b}}{1 - e^{-r/b}} - \frac{1}{b^2} \frac{\alpha(\alpha - 1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{\lambda}{r^2} \right] R(r) = 0, \quad (2.4)$$

$$\Theta''(\theta) + \cot\theta\Theta'(\theta) + \left[-\left(\frac{\beta' + \beta\cos\theta}{\sin^2\theta} \right) + \lambda - \frac{m^2}{\sin^2\theta} \right] \Theta(\theta) = 0 \quad (2.5)$$

respectively.

III. BOUND STATE SOLUTION OF THE RADIAL SCHRÖDINGER EQUATION.

As know, Eq.(2.4) is the radial SE for Manning-Rosen plus a ring-shaped potential. In order to solve Eq.(2.4) with $\lambda = l(l + 1) \neq 0$, we must make an approximation for the centrifugal term. When $r/b \ll 1$, we use an improved approximation scheme [19] to deal with the centrifugal term

$$\left[C_0 + \frac{e^{-r/b}}{(1 - e^{-r/b})^2} \right] \approx \frac{b^2}{r^2} + \left(C_0 - \frac{1}{12} \right) + O\left(\frac{r^2}{b^2}\right), \quad C_0 = \frac{1}{12}, \frac{1}{r^2} \approx \frac{1}{b^2} \left[C_0 + \frac{e^{-r/b}}{(1 - e^{-r/b})^2} \right], \quad (3.1)$$

where the parameter $C_0 = \frac{1}{12}$ is a dimensionless constant. However, when $C_0 = 0$ the approximation scheme becomes the convectional approximation scheme suggested by Greene and Aldrich [20].

We assume $R(r) = \frac{1}{r}\chi(r)$ in Eq.(2.4) and the radial SE becomes

$$\chi''(r) + \left[\frac{2\mu}{\hbar^2}E + \frac{1}{b^2} \frac{Ae^{-r/b}}{(1 - e^{-r/b})} - \frac{1}{b^2} \frac{\alpha(\alpha - 1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{\lambda}{b^2} \left[C_0 + \frac{e^{-r/b}}{(1 - e^{-r/b})^2} \right] \right] \chi(r) = 0. \quad (3.2)$$

In order to transform Eq.(3.2) in to the equation of the generalized hypergeometric-type which is in the form [13]

$$\chi''(s) + \frac{\tilde{\tau}}{\sigma}\chi'(s) + \frac{\tilde{\sigma}}{\sigma^2}\chi(s) = 0, \quad (3.3)$$

we use the transformation $s = e^{-r/b}$. Hence we obtain

$$\chi''(s) + \chi'(s)\frac{1-s}{s(1-s)} + \left[\frac{1}{s(1-s)}\right]^2 \left[-\epsilon^2(1-s)^2 + As(1-s) - \alpha(\alpha-1)s^2 - (1-s)^2\lambda\left(C_0 + \frac{s}{(1-s)^2}\right) \right] \chi(s) = 0, \quad (3.4)$$

where we use the following notation for bound states

$$-\epsilon^2 = \frac{2\mu}{\hbar^2}Eb^2, \quad E < 0. \quad (3.5)$$

Now, we can successfully apply NU method of definition for eigenvalues of energy. By comparing Eq.(3.4) with Eq.(3.3) we can define the followings

$$\begin{aligned} \tilde{\tau}(s) &= 1-s, \sigma(s) = s(1-s), \\ \tilde{\sigma}(s) &= s^2[-\epsilon^2 - A - \alpha(\alpha-1) - \lambda c_0] + s[2\epsilon^2 + A + 2\lambda C_0 - \lambda] + [-\epsilon^2 - \lambda c_0]. \end{aligned} \quad (3.6)$$

If we take the following factorization

$$\chi(s) = \phi(s)y(s), \quad (3.7)$$

for the appropriate function $\phi(s)$ the Eq.(3.4) takes the form of the well known hypergeometric-type equation. The appropriate $\phi(s)$ function must satisfy the following condition

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}, \quad (3.8)$$

where function $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}. \quad (3.9)$$

Finally the equation, where $y(s)$ is one of its solutions, takes the form known as hypergeometric-type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \bar{\lambda}y(s) = 0, \quad (3.10)$$

where

$$\bar{\lambda} = k + \pi', \quad (3.11)$$

and

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \quad (3.12)$$

For our problem, the $\pi(s)$ function is written as

$$\pi(s) = \frac{-s}{2} \pm \sqrt{s^2[a - k] - s[b - k] + c}, \quad (3.13)$$

where the values of the parameters are

$$a = \frac{1}{4} + \epsilon^2 + A + \alpha(\alpha - 1) + \lambda C_0,$$

$$b = 2\epsilon^2 + A + 2\lambda C_0 - \lambda,$$

$$c = \epsilon^2 + \lambda C_0.$$

The constant parameter k can be found complying with the condition that the discriminant of the expression under the square root is equal to zero. Hence we obtain

$$k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a - b)}. \quad (3.14)$$

Now we can find four possible functions for $\pi(s)$

$$\pi(s) = \frac{-s}{2} \pm \begin{cases} (\sqrt{c} - \sqrt{c + a - b})s - \sqrt{c} & \text{for } k = (b - 2c) + 2\sqrt{c^2 + c(a - b)}, \\ (\sqrt{c} + \sqrt{c + a - b})s - \sqrt{c} & \text{for } k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. \end{cases} \quad (3.15)$$

According to NU method, from the four possible forms of the polynomial $\pi(s)$, we select the one for which the function $\tau(s)$ has the negative derivative. Therefore, the appropriate function $\pi(s)$ and $\tau(s)$ are

$$\pi(s) = \sqrt{c} - s \left[\frac{1}{2} + \sqrt{c} + \sqrt{c + a - b} \right], \quad (3.16)$$

$$\tau(s) = 1 + 2\sqrt{c} - 2s \left[1 + \sqrt{c + a - b} \right], \quad (3.17)$$

for

$$k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. \quad (3.18)$$

Also by Eq.(3.11) we can define the constant $\bar{\lambda}$ as

$$\bar{\lambda} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[\frac{1}{2} + \sqrt{c} + \sqrt{c + a - b} \right]. \quad (3.19)$$

Given a nonnegative integer n , the hypergeometric-type equation has a unique polynomials solution of degree n if and only if

$$\bar{\lambda} = \bar{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', (n = 0, 1, 2, \dots), \quad (3.20)$$

and $\bar{\lambda}_m \neq \bar{\lambda}_n$ for $m = 0, 1, 2, \dots, n-1$ [21], then it follows that ,

$$\begin{aligned} \bar{\lambda}_{n_r} &= b - 2c - 2\sqrt{c^2 + c(a-b)} - \left[\frac{1}{2} + \sqrt{c} + \sqrt{c+a-b} \right] \\ &= 2n_r \left[1 + \left(\sqrt{c} + \sqrt{c+a-b} \right) \right] + n_r(n_r - 1). \end{aligned} \quad (3.21)$$

We can solve Eq.(3.21) explicitly for c and by using the relation $c = \epsilon^2 + \lambda C_0$ which brings

$$\epsilon^2 = \left[\frac{\lambda + 1/2 + \Lambda(1 + 2n_r) + n_r(n_r + 1) - A}{2\Lambda + 1 + 2n} \right]^2 - \lambda C_0, \quad (3.22)$$

where $\Lambda = \sqrt{1/4 + \alpha(\alpha - 1) + \bar{\lambda}}$.

We substitute ϵ^2 into Eq.(3.5) with $\lambda = l(l+1)$ which identifies

$$E_{n_r, l} = \frac{-h^2}{2\mu b^2} \left[\left[n_r + 1/2 + \frac{(l - n_r)(l + n_r + 1) - A}{2\Lambda + 1 + 2n_r} \right]^2 - l(l+1)C_0 \right]. \quad (3.23)$$

Now, using NU method we can obtain the radial eigenfunctions. After substituting $\pi(s)$ and $\sigma(s)$ into Eq.(3.8) and solving first order differential equation, it is easy to obtain

$$\phi(s) = s^{\sqrt{c}}(1-s)^K, \quad (3.24)$$

where $K = 1/2 + \Lambda$.

Furthermore, the other part of the wave function $y(s)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)], \quad (3.25)$$

where B_n is a normalizing constant and $\rho(s)$ is the weight function which is the solutions of the Pearson differential equation. The Pearson differential equation and $\rho(s)$ for our problem is given as

$$(\sigma\rho)' = \tau\rho, \quad (3.26)$$

$$\rho(s) = (1-s)^{2K-1} s^{2\sqrt{c}}. \quad (3.27)$$

respectively.

Substituting Eq.(3.27) in Eq.(3.25) we get

$$y_{n_r}(s) = B_{n_r}(1-s)^{1-2k}s^{2\sqrt{c}}\frac{d^{n_r}}{ds^{n_r}}\left[s^{2\sqrt{c}+n_r}(1-s)^{2k-1+n_r}\right]. \quad (3.28)$$

Then by using the following definition of the Jacobi polynomials [22]

$$P_n^{(a,b)}(s) = \frac{(-1)^n}{n!2^n(1-s)^a(1+s)^b}\frac{d^n}{ds^n}\left[(1-s)^{a+n}(1+s)^{b+n}\right], \quad (3.29)$$

we can write

$$P_n^{(a,b)}(1-2s) = \frac{C_n}{s^a(1-s)^b}\frac{d^n}{ds^n}\left[s^{a+n}(1-s)^{b+n}\right], \quad (3.30)$$

and

$$\frac{d^n}{ds^n}\left[s^{a+n}(1-s)^{b+n}\right] = C_n s^a(1-s)^b P_n^{(a,b)}(1-2s). \quad (3.31)$$

If we use the last equality in Eq.(3.28), we can write

$$y_{n_r}(s) = C_{n_r} P_{n_r}^{(2\sqrt{c}, 2K-1)}(1-2s). \quad (3.32)$$

Substituting $\rho(s)$ and $y_{n_r}(s)$ into Eq.(3.7), we obtain

$$\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}}(1-s)^K P_{n_r}^{(2\sqrt{c}, 2K-1)}(1-2s). \quad (3.33)$$

Using the following definition of the Jacobi polynomials [22]

$$P_n^{(a,b)}(s) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)} F_{21}\left(-n, a+b+n+1, 1+a; \frac{1-s}{2}\right), \quad (3.34)$$

we are able to write Eq.(3.33) in terms of hypergeometric polynomials as

$$\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}}(1-s)^K \frac{\Gamma(n_r+2\sqrt{c}+1)}{n_r!\Gamma(2\sqrt{c}+1)} F_{21}\left(-n_r, 2\sqrt{c}+2K+n_r, 1+2\sqrt{c}; s\right). \quad (3.35)$$

The normalization constant C_{n_r} can be found from normalization condition

$$\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = b \int_0^1 \frac{1}{s} |\chi(s)|^2 ds = 1, \quad (3.36)$$

by using the following integral formula [23]

$$\int_0^1 (1-z)^{2(\delta+1)} z^{2\lambda-1} \left\{ F_{21}(-n_r, 2(\delta+\lambda+1) + n_r, 2\lambda+1; z) \right\}^2 dz$$

$$= \frac{(n_r + \delta + 1)n_r! \Gamma(n_r + 2\delta + 2) \Gamma(2\lambda) \Gamma(2\lambda + 1)}{(n_r + \delta + \lambda + 1) \Gamma(n_r + 2\lambda + 1) \Gamma(2(\delta + \lambda + 1) + n_r)}, \quad (3.37)$$

for $\delta > \frac{-3}{2}$ and $\lambda > 0$. After simple calculations, we obtain normalization constant as

$$C_{n_r} = \sqrt{\frac{n_r! 2\sqrt{c}(n_r + K + \sqrt{c}) \Gamma(2(K + \sqrt{c}) + n_r)}{b(n_r + K) \Gamma(n_r + 2\sqrt{c} + 1) \Gamma(n_r + 2K)}}. \quad (3.38)$$

IV. SOLUTION OF AZIMUTHAL ANGLE-DEPENDENT PART OF THE SCHRÖDINGER EQUATION

We may also derive the eigenvalues and eigenvectors of the azimuthal angle dependent part of the SE in Eq.(2.5) by using NU method. The boundary condition for Eq.(2.5), $\Theta(\theta)$ require to be taken as a finite value. Introducing a new variable $x = \cos\theta$, Eq.(2.5) is brought to the form

$$\Theta''(x) - \frac{2x}{1-x^2} \Theta'(x) + \frac{1}{(1-x^2)^2} [\lambda(1-x^2) - m^2 - (\beta' + \beta x)] \Theta(x) = 0. \quad (4.1)$$

After the comparison of Eq.(4.1) with Eq.(3.3) we have

$$\tilde{\tau}(x) = -2x, \quad \sigma(x) = 1 - x^2, \quad \tilde{\sigma}(x) = -\lambda x^2 - \beta x + (\lambda - m^2 - \beta'). \quad (4.2)$$

In the NU method the new function $\pi(x)$ is calculated for angle-dependent part as

$$\pi(x) = \pm \sqrt{x^2(\lambda - k) + \beta x - (\lambda - \beta' - m^2 - k)}. \quad (4.3)$$

The constant parameter k can be determined as

$$k_{1,2} = \frac{2\lambda - m^2 - \beta'}{2} \pm \frac{u}{2}, \quad (4.4)$$

where $u = \sqrt{(m^2 + \beta')^2 - \beta^2}$.

The appropriate function $\pi(x)$ and parameter k are

$$\pi(x) = - \left[x \sqrt{\frac{m^2 + \beta' + u}{2}} + \sqrt{\frac{m^2 + \beta' - u}{2}} \right], \quad (4.5)$$

$$k = \frac{2\lambda - m^2 - \beta'}{2} - \frac{u}{2}. \quad (4.6)$$

The following track in this selection is to achieve the condition $\tau' < 0$. Therefore $\tau(x)$ becomes

$$\tau(x) = -2x \left[1 + \sqrt{\frac{m^2 + \beta' + u}{2}} \right] - 2\sqrt{\frac{m^2 + \beta' - u}{2}}. \quad (4.7)$$

We can also write the values $\bar{\lambda} = k + \pi'(s)$ as

$$\bar{\lambda} = \frac{2\lambda - \beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \beta' + u}{2}}, \quad (4.8)$$

also using Eq.(3.20) we can equate

$$\bar{\lambda}_N = \frac{2\lambda - \beta' - m^2}{2} - \frac{u}{2} - \sqrt{\frac{m^2 + \beta' + u}{2}} = 2N \left[1 + \sqrt{\frac{m^2 + \beta' + u}{2}} \right] + N(N - 1). \quad (4.9)$$

In order to obtain unknown λ we can solve Eq.(4.9) explicitly for $\lambda = l(l + 1)$

$$\lambda - \zeta^2 - \zeta = 2N(1 + \zeta) + N(N - 1), \quad (4.10)$$

where $\zeta = \sqrt{\frac{m^2 + \beta' + u}{2}}$, and

$$\lambda = \zeta^2 + \zeta + 2N\zeta + N(N + 1) = (N + \zeta)(N + \zeta + 1) = l(l + 1), \quad (4.11)$$

then

$$l = N + \zeta. \quad (4.12)$$

Substitution of this result in Eq.(3.23) yields the desired energy spectrum, in terms of n_r and l quantum numbers. Similarly, the wave function of polar angle dependent part of SE can be formally derived by a process to the derivation of radial part of SE.

$$\phi(x) = (1 - x)^{(B+C)/2}, \quad (4.13)$$

$$\rho(x) = (1 - x)^{B+C}(1 + x)^{B-C}, \quad (4.14)$$

$$y_N(x) = B_N(1 - x)^{-(B+C)}(1 + x)^{C-B} \frac{d^N}{dz^N} \left[(1 - x)^{B+C+N}(1 + x)^{B-C+N} \right], \quad (4.15)$$

where

$$B = \sqrt{\frac{m^2 + \beta' + u}{2}}, \quad C = \sqrt{\frac{m^2 + \beta' - u}{2}}.$$

From the definition of Jacobi polynomials, we can write

$$\frac{d^N}{dx^N} [(1-x)^{B+C+N}(1+x)^{B-C+N}] = (-1)^N 2^N (1-x)^{B+C}(1+x)^{B-C} P_N^{(B+C, B-C)}(x). \quad (4.16)$$

Substitution of Eq.(4.16) into Eq.(4.15) and after long but straightforward calculations we obtain the following result

$$\Theta_N(x) = C_N (1-x)^{(B+C)/2} (1+x)^{(B-C)/2} P_N^{(B+C, B-C)}(x), \quad (4.17)$$

where C_N is the normalization constant. Using orthogonality relation of the Jacobi polynomials [22] the normalization constant can be found as

$$C_N = \sqrt{\frac{(2N+2B+1)\Gamma(N+1)\Gamma(N+2B+1)}{2^{2B+1}\Gamma(N+B+C+1)\Gamma(N+B-C+1)}}. \quad (4.18)$$

V. NUMERICAL RESULTS AND DISCUSSION

Solution of the SE for the Manning-Rosen potential plus a ring-shaped like potential are obtained by applying the Nikiforov-Uvarov method in which we used the improved approximation scheme to the centrifugal potential for arbitrary l states. The energy eigenvalues and corresponding eigenfunctions are obtained for arbitrary l quantum numbers. Two important cases must be emphasized in the results of this study. In the first case which $\beta = \beta' = 0$ the potentials turn to central MR potential. For this case, by using $u = m^2$, $\zeta = |m|$ and $l = N + |m|$ ($N = 0, 1, 2, \dots$) then $l \geq |m|$ by substituting this l values in Eq.(3.23) we obtain energy spectrum for MR potential. These results are consistent with those of works in Zhao-You Chen and et al, [24]. Also, if $\alpha = 0$ or $\alpha = 1$, then in this case, for $\delta = 1/b$, $A = 2b$ potential turns to the Hulthén potential with $l \geq |m|$. This energy values are consistent with those of works in Varshni [17]. The second case is the general situation where $\beta \neq 0$ or $\beta' \neq 0$. By changing β and β' we obtain energy values of MR plus different type ring shaped like potentials. In Table 1(for $\alpha = 0.75$) and Table 2(for $\alpha = 1$) we show energies of the bound states for the Manning-Rosen potential plus ring shaped potentials for different

values of β, β', m, l, n and $1/b = 0.025, A = 2b$, where $n = n_r + l + 1$, is the usual principle quantum number and n_r is the number of nodes of the radial wave functions. To show accuracy of our results, the plots of the centrifugal term $1/r^2$ (solid line), the improved new approximation to it $\frac{1}{r^2} = \frac{1}{b^2}[C_0 + \frac{e^{-r/b}}{(1-e^{-r/b})^2}]$ (dashed) and the conventional approximation to it $\frac{1}{r^2} = \frac{1}{b^2}[\frac{e^{-r/b}}{(1-e^{-r/b})^2}]$ (long dashed) as function of the variable r are displayed from figures 1 to 3 with different potential range parameter $b=1,2,4$. It is shown that the our approximation Eq.(3.1) is a good approximation to the centrifugal term for short range potential range, i.e. large b .

Finally we want to deal with some restrictions about bound state solutions of SE for Manning Rosen plus ring shaped like potential. Firstly, it is seen from Eq(4.4) and expression from u that in order to obtain real energy values the condition $(m^2 + 2\beta')^2 \geq 4\beta^2$ must be hold. Since the parameters β and β' are real and positive, we can write

$$m^2 \geq 2(\beta - \beta'). \quad (5.1)$$

If $\beta \leq \beta'$ the inequality in Eq.(5.1) is provided automatically. But if $\beta \geq \beta'$ then m becomes bounded. Secondly, in Eq.(3.23) if

$$l(l+1)C_0 < \left[n_r + 1/2 + \frac{(l - n_r)(l + n_r + 1) - A}{2\Lambda + 1 + 2n_r} \right]^2, \quad (5.2)$$

then energy eigenvalues take nonnegative values, this means there is no bound states. If we take $C_0 = 0$ for the approaches in centrifugal term as some previous studies, the restriction on quantum numbers is removed. Finally from Eq (3.21) we obtain

$$\sqrt{c} = \frac{\lambda + 1/2 + \Lambda(1 + 2n_r) + n_r(n_r + 1) - A}{-(2\Lambda + 1 + 2n_r)}. \quad (5.3)$$

For bound states since $c > 0$ (so $\sqrt{c} > 0$) and using the fact that $2\Lambda + 1 + 2n_r > 0$, we obtain

$$\begin{aligned} A - 1/2 - l(l+1) - \Lambda &> n_r(n_r + 2\Lambda + 1), \\ A &> \frac{1}{2} + l(l+1) + \Lambda. \end{aligned} \quad (5.4)$$

If both conditions in Eqs.(5.1-5.2 and 5.4) are satisfied simultaneously, the bound states exist. Thus the energy spectrum equation in Eq.(3.23) as limited, i.e, we have only the finite numbers of energy eigenvalues. Figure 4 and Figure 5 shows the region of possible values of n_r and l quantum numbers according to our analysis in Eq.(5.2) and Eq.(5.4) for $A = 80, \alpha = 1$ and $\alpha = 0.75$ respectively.

VI. CONCLUSION

Analytical calculations of energy eigenvalues for an arbitrary l state and corresponding eigenfunctions in the Manning-Rosen potential plus a Ring-Shaped like potential is done by using Nikiforov-Uvarov Method in this paper. The energy eigenvalue expression for Manning-Rosen potential plus a Ring-Shaped like potential is given by Eq.(3.23). By some specific value of α , β and β' parameters one can observe that the some results of previous studies for finding MR and Hulthen potential can be found. We also obtain some important restrictions on quantum numbers about bound state solutions of SE. We can conclude that our results are not only interesting for pure theoretical physicist but also for experimental physicist, because the results are exact and more general. We have also examined that, obtained results are different from results given in [18]. Therefore, calculation in [18] should be checked and recalculated and compared with our results and [17, 24].

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β	β'	m	N	n_r	l	n	E
0	0	0	0	0	0	1	-0.872300
0	0	0	1	0	0	2	-0.150269
0	0	1	0	0	1	2	-0.120527
1	1	0	0	0	1	2	-0.112760
0	1	0	0	0	1.414214	2.414214	-0.076883
1	1	1	0	0	1.618034	2.618034	-0.063090
0	1	1	0	0	1.732051	2.732051	-0.056764
0	0	0	2	0	0	3	-0.053926
1	0	2	0	0	1.931852	2.931852	-0.047497
0	0	1	1	0	1	3	-0.045878
1	1	0	1	0	1	3	-0.045878
0	0	0	0	2	2	3	-0.044774
1	1	0	0	1	2	3	-0.044774
0	1	0	1	0	1.414214	3.414214	-0.032286
1	1	2	0	0	2.414214	3.414214	-0.031721
0	1	0	0	1	2.414214	3.414214	-0.031721
0	1	2	0	0	2.449490	3.449490	-0.030831
1	1	1	1	0	1.618034	3.618034	-0.027392
1	1	1	0	1	2.618034	3.618034	-0.026949
0	1	1	1	0	1.732051	3.732051	-0.025026

β	β'	m	N	n_r	l	n	E
0	1	1	0	1	2.732051	3.732051	-0.024630
0	0	0	3	0	0	4	-0.024017
1	0	2	1	0	1.931852	3.931852	-0.021403
1	0	2	0	1	2.931852	3.931852	-0.021065
0	0	1	2	0	1	4	-0.020809
1	1	0	2	0	1	4	-0.020809
0	0	0	1	2	2	4	-0.020299
1	1	0	1	1	2	4	-0.020299
1	0	3	0	0	2.981188	3.981188	-0.020271
0	0	1	0	2	3	4	-0.019976
1	1	0	0	2	3	4	-0.019976
1	1	3	0	0	3.302776	4.302776	-0.015781
0	1	3	0	0	3.316625	4.316625	-0.015611
0	1	0	2	0	1.414214	4.414214	-0.015053
0	1	0	1	1	2.414214	4.414214	-0.014733
1	1	2	1	0	2.414214	4.414214	-0.014733
1	1	2	0	1	3.414214	4.414214	-0.014463
0	1	0	0	2	3.414214	4.414214	-0.014463
0	1	2	1	0	2.449490	4.449490	-0.014336

TABLE I: Energies of the bound states for the Manning Rosen potentials plus ring shaped potentials for different values of β, β', m, l, n_r and $\alpha = 0.75, 1/b = 0.025, A = 2b$ calculated using Eq.(4.12) and Eq.(3.23) for $\mu = \hbar = 1$.

β	β'	m	N	n_r	l	n	E
0	0	0	0	0	0	1	-0.487578
0	0	0	1	0	0	2	-0.112812
0	0	1	0	0	1	2	-0.112760
1	1	0	0	0	1	2	-0.112760
0	1	0	0	0	1.414214	2.414214	-0.073653
1	1	1	0	0	1.618034	2.618034	-0.060874
0	1	1	0	0	1.732051	2.732051	-0.054947
1	0	2	0	0	1.931852	2.931852	-0.046192
0	0	0	2	0	0	3	-0.043759
0	0	1	1	0	1	3	-0.043707
1	1	0	1	0	1	3	-0.043707
0	0	2	0	2	2	3	-0.043602
1	1	0	0	1	2	3	-0.043602
0	1	0	1	0	1.414214	3.414214	-0.031215
1	1	2	0	0	2.414214	3.414214	-0.031089
0	1	0	0	1	2.414214	3.414214	-0.031089
0	1	2	0	0	2.449490	3.449490	-0.030230
1	1	1	1	0	1.618034	3.618034	-0.026609
1	1	1	0	1	2.618034	3.618034	-0.026473
0	1	1	1	0	1.732051	3.732051	-0.024363

β	β'	m	N	n_r	l	n	E
0	1	1	0	1	2.732051	3.732051	-0.024221
1	0	2	1	0	1.931852	3.931852	-0.020903
1	0	2	0	1	2.931852	3.931852	-0.020750
0	0	0	3	0	0	4	-0.020000
1	0	3	0	0	2.981188	3.981188	-0.019975
0	0	1	2	0	1	4	-0.019948
1	1	0	2	0	1	4	-0.019948
1	1	0	1	1	2	4	-0.019844
0	0	1	1	1	2	4	-0.019844
0	0	3	0	0	3	4	-0.019687
1	1	0	0	2	3	4	-0.019687
1	1	3	0	0	3.302776	4.302776	-0.015583
0	1	3	0	0	3.316625	4.316625	-0.015417
0	1	0	2	0	1.414214	4.414214	-0.014594
0	1	0	1	1	2.414214	4.414214	-0.014468
1	1	2	1	0	2.414214	4.414214	-0.014468
1	1	2	0	1	3.414214	4.414214	-0.014290
0	1	0	0	2	3.414214	4.414214	-0.014290
0	1	2	1	0	2.449490	4.449490	-0.014082

TABLE II: Energies of the bound states for the Manning Rosen potentials plus ring shaped potentials for different values of β, β', m, l, n_r and $\alpha = 1, 1/b = 0.025, A = 2b$ calculated using Eq.(4.12) and Eq.(3.23) for $\mu = \hbar = 1$.

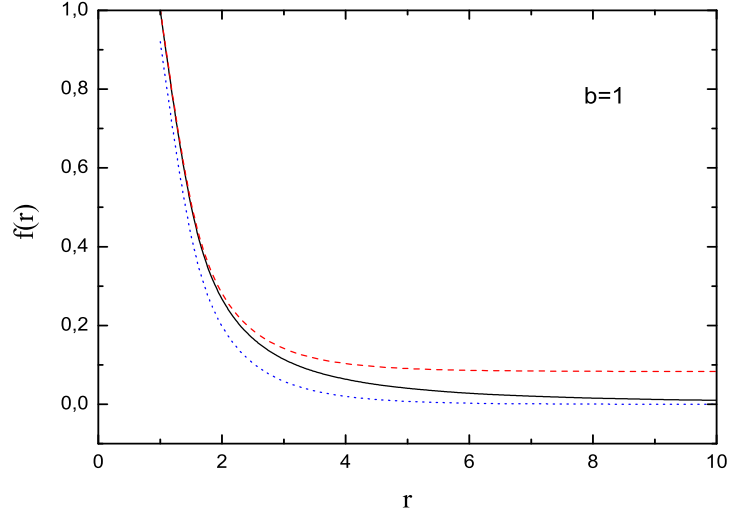


FIG. 1: The plots of the centrifugal term (solid line) the improved new approximation to it (dashed) and the conventional approximation to it (long dashed) as the function of the variable r with potential range parameter $b=1$

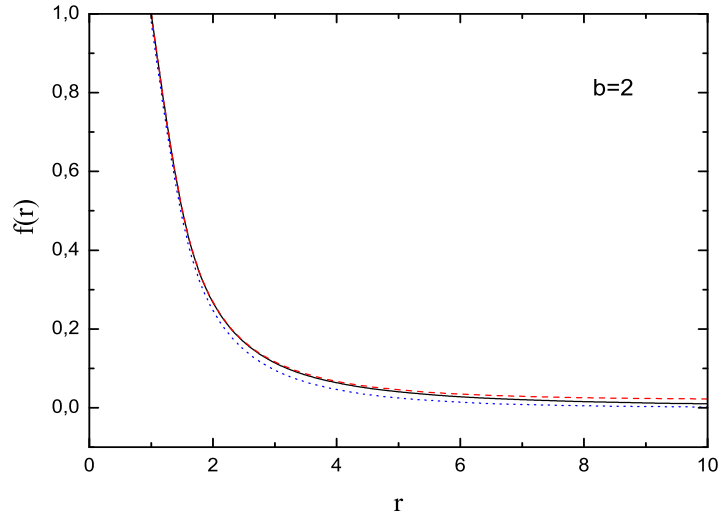


FIG. 2: The plots of the centrifugal term (solid line) the improved new approximation to it (dashed) and the conventional approximation to it (long dashed) as the function of the variable r with potential range parameter $b=2$

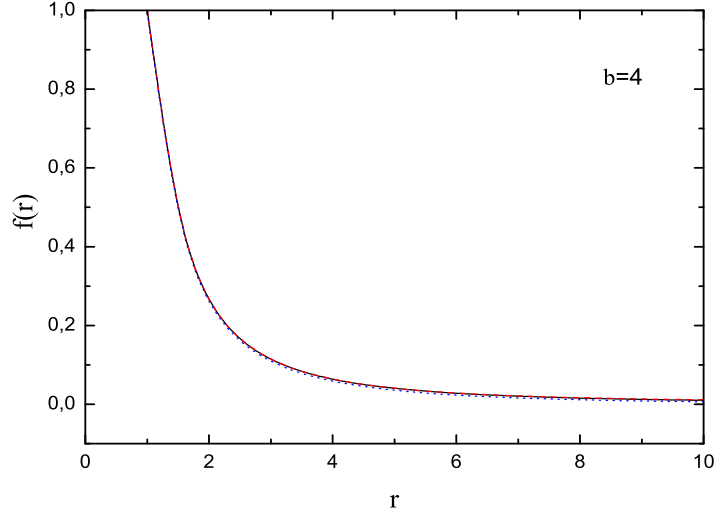


FIG. 3: The plots of the centrifugal term (solid line) the improved new approximation to it (dashed) and the conventional approximation to it (long dashed) as the function of the variable r with potential range parameter $b=4$

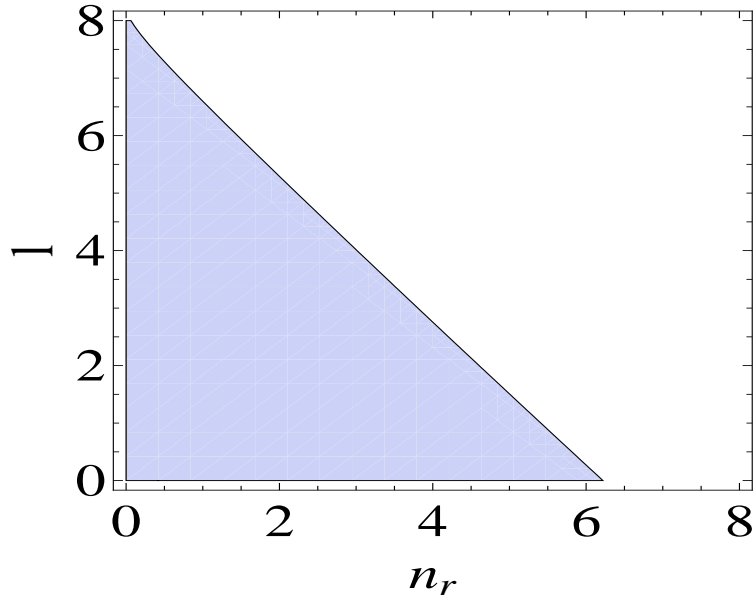


FIG. 4: The region of possible values of n_r and l for $\alpha = 0.75$, $1/b=0.025$ and $A=2b$

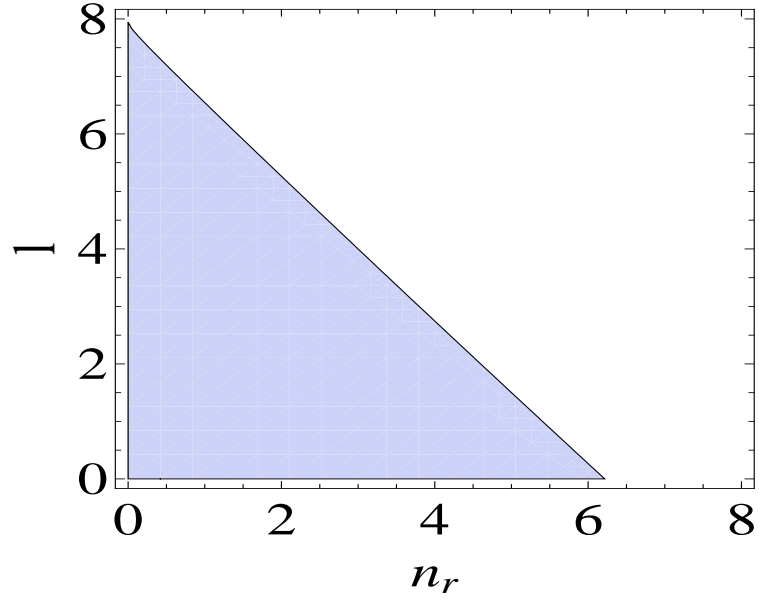


FIG. 5: The region of possible values of n_r and l for $\alpha = 1$, $1/b=0.025$ and $A=2b$